Simplified Derivation of the Heston Model

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Note: A complete treatment of the Heston model, including a more detailed derivation of what appears below, is available in the forthcoming book "The Heston Model and its Extensions in Matlab and C#", available September 3, 2013 from John Wiley & Sons. The table of contents is available at www.FRouah.com.

The stochastic volatility model of Heston [2] is one of the most popular equity option pricing models. This is due in part to the fact that the Heston model produces call prices that are in closed form, up to an integral that must evaluated numerically. In this Note we present a complete derivation of the Heston model.

1 Heston Dynamics

The Heston model assumes that the underlying, S_t, follows a Black-Scholes type stochastic process, but with a stochastic variance v_t that follows a Cox, Ingersoll, Ross process. Hence

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_{1,t} \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t} \\ E\left[dW_{1,t} dW_{2,t}\right] &= \rho dt. \end{aligned} \tag{1}$$

We will drop the time index and write $S = S_t, v = v_t, W_1 = W_{1,t}$, and $W_2 = W_{1,t}$ for notional convenience.

2 The Heston PDE

In this section we explain how to derive the PDE from the Heston model. This derivation is a special case of a PDE for general stochastic volatility models which is described by Gatheral [1]. Form a portfolio consisting of one option $V = V(S, v, t)$, Δ units of the stock S, and ϕ units of another option $U = U(S, v, t)$ that is used to hedge the volatility. The portfolio has value

$$\Pi = V + \Delta S + \phi U$$

where $\Pi = \Pi_t$. Assuming the portfolio is self-financing, the change in portfolio value is

$$d\Pi = dV + \Delta dS + \phi dU.$$

2.1 Portfolio Dynamics

Apply Itō's Lemma to dV. We must differentiate with respect to the variables t, S, and v. Hence

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}dt + \sigma v\rho S\frac{\partial^2 V}{\partial v \partial S}dt.$$

Applying Itō's Lemma to dU produces the identical result, but in U. Combining these two expressions, we can write the change in portfolio value, $d\Pi$, as

$$\begin{aligned} d\Pi &= dV + \Delta dS + \phi dU \qquad (2) \\ &= \left\{\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}\right\}dt + \\ &\quad \phi\left\{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right\}dt + \\ &\quad \left\{\frac{\partial V}{\partial S} + \phi\frac{\partial U}{\partial S} + \Delta\right\}dS + \left\{\frac{\partial V}{\partial v} + \phi\frac{\partial U}{\partial v}\right\}dv. \end{aligned}$$

2.2 The Riskless Portfolio

In order for the portfolio to be hedged against movements in the stock and against volatility, the last two terms in Equation (2) involving dS and dv must be zero. This implies that the hedge parameters must be

$$\begin{aligned} \phi &= -\frac{\frac{\partial V}{\partial v}}{\frac{\partial U}{\partial v}} \qquad (3) \\ \Delta &= -\phi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}. \end{aligned}$$

Moreover, the portfolio must earn the risk free rate, r. Hence $d\Pi = r\Pi dt$. Now with the values of ϕ and Δ from Equation (3) the change in value of the riskless portfolio is

$$\begin{aligned} d\Pi &= \left\{\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}\right\}dt + \\ &\quad \phi\left\{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right\}dt \end{aligned}$$

which we write as $d\Pi = (A + \phi B)\,dt$. Hence we have

$$A + \phi B = r\left(V + \Delta S + \phi U\right).$$

Substituting for ϕ and re-arranging, produces the equality

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} = \frac{B - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}} \qquad (4)$$

which we exploit in the next section.

2.3 The PDE in Terms of the Price

The left-hand side of Equation (4) is a function of V only, and the right-hand side is a function of U only. This implies that both sides can be written as a function $f(S, v, t)$ of S, v, and t. Following Heston, specify this function as $f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t)$, where $\lambda(S, v, t)$ is the price of volatility risk. Substitute $f(S, v, t)$ into the left-hand side of Equation (4), substitute for B, and rearrange to produce the Heston PDE expressed in terms of the price S

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v S\frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} \\ &-rU + rS\frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda(S, v, t)]\frac{\partial U}{\partial v} = 0. \end{aligned} \tag{5}$$

This is Equation (6) of Heston [2].

2.4 The PDE in Terms of the Log Price

Let $x = \ln S$ and express the PDE in terms of x, t and v instead of S, t, and v. This leads to a simpler form of the PDE. We need the following derivatives, which are straightforward to derive

$$\frac{\partial U}{\partial S}, \frac{\partial^2 U}{\partial v \partial S}, \frac{\partial^2 U}{\partial S^2}$$

Plug into the Heston PDE Equation (5). All the S terms cancel and we obtain the Heston PDE in terms of the log price $x = \ln S$

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}v\right)\frac{\partial U}{\partial x} + \rho\sigma v\frac{\partial^2 U}{\partial v \partial x} + \\ &\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} - rU + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} = 0 \end{aligned} \tag{6}$$

where, as in Heston, we have written the market price of risk to be a linear function of the volatility, so that $\lambda(S, v, t) = \lambda v$.

3 The Call Price

The call price is of the form

$$\begin{aligned} C_T(K) &= e^{-r\tau}E\left[(S_T - K)^+\right] \\ &= e^{x_t}P_1(x, v, \tau) - e^{-r\tau}KP_2(x, v, \tau). \end{aligned} \tag{7}$$

In this expression $P_j(x, v, \tau)$ each represent the probability of the call expiring in-the-money, conditional on the value $x_t = \ln S_t$ of the stock and on the value v_t of the volatility at time t, where $\tau = T - t$ is the time to expiration. Now take the following derivatives of C using (7). These are straightforward to obtain

$$\frac{\partial C}{\partial t}, \frac{\partial C}{\partial x}, \frac{\partial^2 C}{\partial x^2}, \frac{\partial C}{\partial v}, \frac{\partial^2 C}{\partial v^2}, \frac{\partial^2 C}{\partial x \partial v}.$$

We use these derivatives in the following section.

3.1 The PDE for P_1 and P_2

The call price C in Equation (7) follows the PDE in Equation (6), which we write here in terms of C but using the time derivative with respect to τ rather than t

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2}v\frac{\partial^2 C}{\partial x^2} + \left(r - \frac{1}{2}v\right)\frac{\partial C}{\partial x} + \rho\sigma v\frac{\partial^2 C}{\partial v \partial x} + \tag{8}$$
$$\frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} - rC + [\kappa(\theta - v) - \lambda v]\frac{\partial C}{\partial v} = 0.$$

The derivatives of C from (7) will be in terms of P_1 and P_2. Substitute these derivatives into the PDE (8) and regroup terms common to P_1. Set $K = 0$ and $S = 1$ to obtain the PDE for P_1. Now regroup terms common to P_2 and set $S = 0, K = -1$, and $r = 0$ to obtain the PDE for P_2 For notional convenience, combine the PDEs for P_1 and P_2 into a single expression

$$-\frac{\partial P_j}{\partial \tau} + \rho\sigma v\frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2}v\sigma^2\frac{\partial^2 P_j}{\partial v^2} \tag{9}$$
$$+ (r + u_j v)\frac{\partial P_j}{\partial x} + (a - b_j v)\frac{\partial P_j}{\partial v} = 0$$

for $j = 1, 2$ and where $u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa + \lambda - \rho\sigma$, and $b_2 = \kappa + \lambda$. This is Equation (12) of Heston [2] but in terms of τ rather than t. That explains the minus sign in the first term of Equation (9) above.

3.2 Obtaining the Characteristic Functions

Heston assumes that the characteristic functions for the logarithm of the terminal stock price, $x = \ln S_T$, are of the form

$$f_j(\phi; x, v) = \exp\left(C_j(\tau, \phi) + D_j(\tau, \phi)\, v_0 + i\phi x\right) \tag{10}$$

where C_j and D_j are coefficients and $\tau = T - t$ is the time to maturity. The characteristic functions f_j will follow the PDE in Equation (9). This is a consequence of the Feynman-Kac theorem. Hence the PDE for the characteristic function is, from Equation (9)

$$-\frac{\partial f_j}{\partial \tau} + \rho\sigma v\frac{\partial^2 f_j}{\partial x \partial v} + \frac{1}{2}v\frac{\partial^2 f_j}{\partial x^2} + \frac{1}{2}v\sigma^2\frac{\partial^2 f_j}{\partial v^2} \tag{11}$$
$$+ (r + u_j v)\frac{\partial f_j}{\partial x} + (a - b_j v)\frac{\partial f_j}{\partial v} = 0.$$

To evaluate this PDE for the characteristic function we need the following derivatives, which are straightforward to derive

$$\frac{\partial f_j}{\partial \tau}, \frac{\partial f_j}{\partial x}, \frac{\partial^2 f_j}{\partial x^2}, \frac{\partial f_j}{\partial v}, \frac{\partial^2 f_j}{\partial v^2}, \frac{\partial^2 f_j}{\partial v \partial x}.$$

Substitute these derivatives in Equation (11), drop the f_j terms, and re-arrange to obtain two differential equations

$$\begin{aligned}\frac{\partial D_j}{\partial \tau} &= \rho\sigma i\phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j \\ \frac{\partial C_j}{\partial \tau} &= ri\phi + aD_j.\end{aligned} \tag{12}$$

These are Equations (A7) in Heston [2]. Heston specifies the initial conditions $D_j(0,\phi) = 0$ and $C_j(0,\phi) = 0$. The first Equation in (12) is a Riccati equation in D_j while the second is an ODE for C_j that can solved using straightforward integration once D_j is obtained.

3.3 Solving the Heston Riccati Equation

From Equation (12) the Heston Riccati equation is

$$\frac{\partial D_j}{\partial \tau} = P_j - Q_j D_j + RD_j^2 \tag{13}$$

The corresponding second order ODE is

$$w'' + Q_j w' + P_j R = 0 \tag{14}$$

The solution to the Heston Riccati equation (13) is therefore

$$D_j = -\frac{1}{R}\left(\frac{K\alpha e^{\alpha\tau} + \beta e^{\beta\tau}}{Ke^{\alpha\tau} + e^{\beta\tau}}\right) \tag{15}$$

Using the initial condition $D_j(0,\phi) = 0$ produces the solution for D_j

$$D_j = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right). \tag{16}$$

where

$$d_j = \sqrt{\left(\rho\sigma i\phi - b_j\right)^2 - \sigma^2\left(2u_j i\phi - \phi^2\right)}.$$

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}.$$

The solution for C_j is found by integrating the second equation in (12). Hence

$$C_j = \int_0^\tau ri\phi dy + a\left(\frac{Q_j + d_j}{2R}\right)\int_0^\tau\left(\frac{1 - e^{d_j y}}{1 - g_j e^{d_j y}}\right)dy + K_1 \tag{17}$$

where K_1 is a constant. Integrate and apply the initial condition $C_j(0,\phi) = 0$, and substitute for d_j, Q_j, and g_j to obtain the solution for C_j

$$C_j = ri\phi\tau + \frac{a}{\sigma^2}\left[\left(b_j - \rho\sigma i\phi + d_j\right)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right]. \tag{18}$$

where $a = \kappa\theta$.

References

[1] Gatheral, J. (2006) *The Volatility Surface: A Practitioner's Guide*. New York, NY: John Wiley & Sons.

[2] Heston, S.L. (1993). "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options." *Review of Financial Studies*, Vol. 6, pp 327-343.

[3] Lewis, A.L.(2000). *Option Valuation Under Stochastic Volatility: With Mathematica Code*. Finance Press.